INT. J. PROD. RES., 2000, VOL. 38, NO. 7, 1631–1639

Production outsourcing: a linear programming model for the Theory-Of-Constraints

ALEX COMAN† and BOAZ RONEN‡*

This paper presents an analysis of the outsourcing problem. Pertinent variables are identified and the relationships between them are defined. We formulate the outsourcing problem as a Linear-Programming (LP) problem and identify an analytical solution. We proceed with an example examining three decision models: standard cost accounting, standard Theory-Of-Constraints (TOC) and our own solution. The model enables managers to determine which products to manufacture and which to outsource. The solution of the LP formulation enables managers to apply the model by computing an operational ratio, without having to solve a linear programming problem. The final model is simpler to apply and requires the computation of fewer variables than other prevalent models.

1. Introduction

Manufacturing companies often function in situations where internal production resources constrain their throughput. Such situations are characterized by market demand in excess of the company's production capacity. This problem, of finite capacity scheduling, is treated throughout the literature. This study examines the case where market demand exceeds the company's capacity to manufacture. Management needs to decide what quantities of each product to manufacture and what quantities to buy from external contractors.

Information pertinent to the production versus outsourcing problem apparently includes the cost of raw materials, the company's hourly rate, the product's sale price, total machine time dedicated to the product, work time at the bottleneck, product flow through the different resources etc.

Since different models provide radically different answers to the outsourcing problem we compare three alternatives: standard cost accounting, standard Theory-Of-Constraints (TOC) (Goldratt 1988, 1991, Fox 1988, Ronen and Starr 1990), and our Linear Programming (LP) enhancement of the TOC.

We start with a formal presentation of the outsourcing problem, continue with our LP analysis of the problem and proceed to present an example illustrating the different outcomes of the three models.

The TOC offers a five step Constraint-Management-Cycle (CMC) (Coman and Ronen 1995) methodology for the identification of organizational constraints and their elevation (Ronen and Starr 1990, Floyd and Ronen 1989). The methodology is described as follows.

Revision received August 1999.

† Faculty of Technology Management, Center for Technology Education Holon, Israel.
‡ Faculty of Management, Tel-Aviv University, Tel-Aviv, 69978, Israel.
* To whom correspondence should be addressed. e-mail: boaz1@post.tau.ac.il

International Journal of Production Research ISSN 0020–7543 print/ISSN 1366–588X online © 2000 Taylor & Francis Ltd
http://www.tandf.co.uk/journals/tf/00207543.html

Step 1. Identify the system's constraints.
Step 2. Decide how to exploit the constraints.
Step 3. Subordinate everything else to the above decision.
Step 4. Elevate the system constraints.
Step 5. If, in the previous steps, the constraint was violated, go back to step 3, but do not let inertia become the system's constraint.

This paper considers the sixth step: elevate the system constraints, in terms of extending capacity through usage of outside contractors.

2. The outsourcing problem

Consider a production facility consisting of N resources $(1, 2, \ldots, j, \ldots, N)$, manufacturing M different products $(1, 2, \ldots, i, \ldots, M)$. Let b_j denote resource j's capacity in working minutes per week, and let a_{ij} denote the number of minutes required by resource j to process product i.

The cost of raw materials incorporated in product i is denoted by RM_i, and the product's market price is MP_i. The market demand quantity for product i at these prices is denoted by DQ_i.

Due to its capacity constraint, the company can only manufacture X_i units of product i, where $X_i \leq DQ_i$. It is company policy to supply all the market demand in order to prevent other major competitors from penetrating the market, and at the same time maintain the company s reputation for due-date-performance. The company therefore commissions outside contractors to fill in the gap between the quantities that it produces and the market demand. Contractors purchase their own raw materials and deliver product i at the price of CP_i.

The company's additional expenses, such as labour, energy and financing are treated as Operating Expenses (OE).

3. Linear programming analysis

Management aims to maximize its throughput from manufacturing and from outsourcing products. The throughput from manufacturing one unit of product i is defined as the difference between market price and the cost of raw materials, or: $MP_i - RM_i$. Production of X_i units of product i generates a total throughput of $X_i(MP_i - RM_i)$. To satisfy market demand the company orders a quantity of $DQ_i - X_i$ from outside contractors, generating a throughput of $MP_i - CP_i$ per contracted unit, accounting for a total throughput of $(DQ_i - X_i)(MP_i - CP_i)$ for product i, units contracted outside.

Thus total profits from product i equal:

$$X_i(Mp_i - RM_i) + (DQ_i - X_i)(MP_i - CP_i).$$

Profits from all M products total:

$$\sum_{i-1}^{M}[X_i(MP_i - RM_i) + (DQ_i - X_i)(MP_i - CP_i)] - OE$$

where OE denotes the company's operating expenses.

The total throughput from the plant is restricted by its production capacity at resource j as well as by the market demand for product i. (The TOC assumption is that the LP solution does not have more than one binding constraint.)

The product mix problem can now be expressed in standard linear programming format as:

$$\text{Max} \sum_{i=1}^{M} [X_i(MP_i - RM_i) + (DQ_i - X_i)(MP_i - CP_i)] - OE$$

Subject to:

$$X_i \leq DQ_i; \quad i \in \{1, \dots, M\} \text{ Product demand constraints.}$$

$$\sum_{i=1}^{M} a_{ij} X_i \leq b_j; \quad j \in \{1, \dots, N\} \text{ Manufacturing resource constraints.}$$

Analysis of the objective function demonstrates that it can be simplified as follows:

$$\sum_{i=1}^{M} [(MP_i - RM_i)X_i + (MP_i - CP_i)(DQ_i - X_i)] - OE$$

$$= \sum_{i=1}^{M} (MP_i X_i - RM_i X_i + MP_i DQ_i - CP_i DQ_i + CP_i X_i - MP_i X_i) - OE$$

$$= \sum_{i-1}^{M} (CP_i - RM_i)X_i + (MP_i - CP_i) \times DQ_i - OE$$

$$= \sum_{i-1}^{M} (CP_i - RM_i)X_i + \text{Const.} = \sum_{i=1}^{M} (\text{Contractor markup})X_i + \text{Constant.}$$

The new simplified formulation of the LP problem ignores the constant expressions in the objective function:

$$\text{Max} \sum_{i=1}^{M} X_i(CP_i - RM_i)(*)$$

Subject to:

$$\sum_{i-1}^{M} a_{ij} X_i \leq b_j; \quad j \in \{1, \dots, M\} \text{ Manufacturing resource constraints.}$$

$$X_i \leq DQ_i; \quad i \in \{1, \dots, M\} \text{ Product demand constraints.}$$

The implications of the last expression (*) are that product price, cost of raw materials, total working hours per product or hourly rate are totally irrelevant to the outsourcing decision.

Only two variables are relevant:

(1) Contractor markup per product: $CP_i - RM_i$; and
(2) Time per product at the bottleneck resource, j: a_{ij}.

This LP formulation need not be solved for each situation because in many plants there is a single resource constraining the capacity of the whole facility. Now, we can prioritize the products for production and for outsourcing. The order is determined by the ratio of contractor markup per bottleneck minutes:

$$(CP_i - Rm_i)/a_{ij} = \text{(Contractor Price – Raw Materials)/Constraint minutes;}$$

where j = constrained resource (bottleneck).

The implication of this ratio is that products whose outside contractors are greedy (in terms of markup per constraint minutes) are of the highest priority to manufacture in-house, and the stronger the incentive to manufacture in-house, and the less greedy the contractor, the stronger the incentive to outsource.

Thus, whenever outside contractors are available, the standard Throughput-per-Constraint-Time ratio used by the TOC is replaced by the Contractor-Markup-per-Constraint-Time.

This theoretical development eliminates the need to resolve an LP problem for each application of the model. Instead, it is enough to order products by the above ratio and schedule them one by one to manufacture the market demand until the bottleneck resource is assigned to its full capacity.

The LP model expands the TOC body of knowledge by stating that when outsourcing is a true alternative, the throughput of the manufacturing facility should be the alternative price minus the raw material cost.

The following example illustrates the differences between standard accounting, standard TOC and our LP model in terms of pertinent variables, mode of analysis, and total throughput.

4. An example of the outsourcing problem

Consider a production facility consisting of four resources: A, B, C and D, manufacturing three different products: P, Q and R. The facility operates five days a week for an eight hour shift per day. Capacity thus totals 2400 working minutes per week. Table 1 describes the standard time allocated by each resource to every product in minutes.

As illustrated in table 1, satisfying the 100 unit demand level requires 4200 minutes of resource B per week, which constitute 175% of the resource's capacity. Resource B is thus the bottleneck constraining the facility's capacity. Management policy is to meet all demand in order to prevent competitors from entering the arena.

Figure 1 depicts the flow layout throughout the production floor. Each of the three products incorporates two of four raw materials: 1, 2, 3 and 4. The cost of each raw material unit is $20, thus the total value of raw materials in each final product unit is $40. The products are sold on the market at $130 for unit of P, $150 for unit of Q, and $190 for unit of R. The market demand is 100 units of each one of the three products.

Contractors supply product P for $66, product Q for $68 and product R for $98. These prices include the cost of raw materials.

Resource	Product P	Product Q	Product R	Minutes/week: for 3×100 units	Utilization ratio: demand/2400 min
A	2	4	13	1900	79%
B	12	12	18	4200	175%
C	4	10	10	2400	100%
D	4	6	10	2000	83%
Total min/prod.	22	32	51		

Table 1. Resource time per product in minutes.

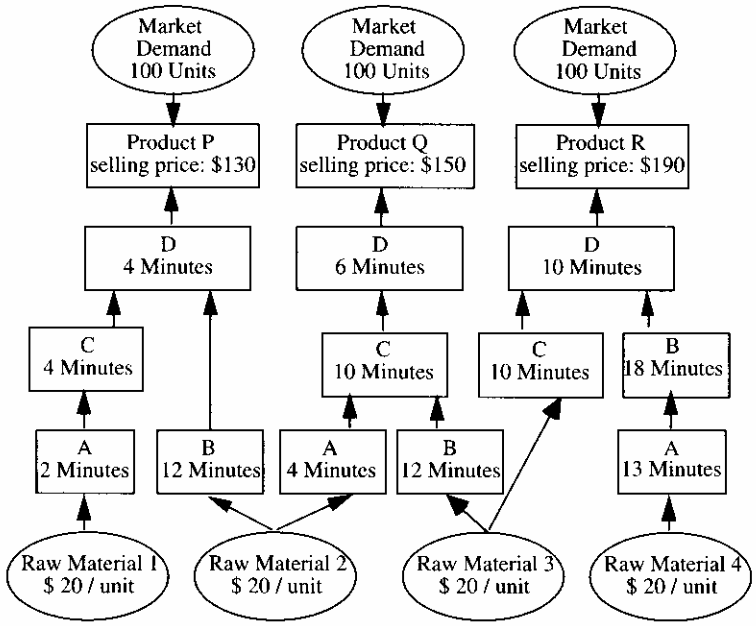

Figure 1. Resource time per product in minutes.

The company's operating expenses of \$12 000 include labour, energy, financing etc.

We start by analysing the above problem according to standard accounting practices. Next we analyse the problem according to standard TOC practices. Finally we analyse the problem using the LP analysis.

5. Standard accounting

Standard accounting procedure gives preference to those products that are more profitable per production time unit. To compute the profitability of each product, the company's operating expenses of \$12 000 are divided by the number of resources: four, and by the work minutes per week: 2400. Thus, the cost of every minute worked at any resource equals 12 000/4/2400 = \$1.25.

The cost of each product (table 2) is the sum of its raw materials (\$40), plus its total working minutes at all four resources, multiplied by the company's minute rate (\$1.25). The profit per product *is* computed by subtracting its cost from its market price. The product's profit is divided by the total work time in the facility to determine its profit per work minute: \$2.84 for product P, \$2.19 for product Q, and \$1.69 for product R. P is the most attractive product to manufacture since its profit per work minute is the highest, followed by Q, making R the least attractive to manufacture. Resource B will therefore manufacture 100 units of product P, 100 units of product Q, but no units of product R—due to its time availability constraint. Product R will have to be subcontracted completely—100 units.

We proceed to compute the throughput of this product mix solution. The throughput of each manufactured unit (table 2) is the market price less the cost of raw materials. This figure is multiplied by the number of units manufactured. The throughput of each unit contracted outside is computed as its market price less the

	Product P	Product Q	Product R
Total work minutes/product unit	22	32	51
Product cost: time + raw materials	$68	$80	$104
Product market price	$130	$150	$190
Product profit: price-cost	$63	$70	$86
Product profit per work minute	$2.84	$2.19	$1.69
Demand in units	100	100	100
Units to manufacture	100	100	0
Throughput/manufactured unit	$90	$110	$150
Units contracted outside	0	0	100
Throughput/contracted unit	$64	$82	$92
Total product throughput	$9000	$11 000	$9200
Total facility throughput		$29 200	
Operating expenses		$12 000	
Net profit		$17 200	

Table 2. Standard accounting analysis.

price paid to the contractor. Total product throughput is the sum of its manufactured and contracted throughputs. From the total facility throughput of $29 200 we subtract its operating expenses of $12 000 generating a net profit of $17 200.

We next examine the standard TOC approach to the outsourcing problem.

6. Standard Theory-Of-Constraints

he Theory-Of-Constraints (TOC) rates the attractiveness of manufacturing a product based upon its throughput per unit of working time at the bottleneck. A product's throughput is defined as the cost of raw materials subtracted from its sale price. We divide the throughput by the time invested by the bottleneck resource B in the product. Product Q has the highest throughput per constraint minutes ($9.17/constraint minute) and is therefore the first preference to manufacture. We thus

	Product P	Product Q	Product R
Market price per unit	$130	$150	$190
Raw material cost per unit	$40	$40	$40
Throughput per unit	$90	$110	$150
Constraint resource B min/unit	12	12	18
Throughput per constraint minutes	$90/12 min = $7.50/min	$110/12 min = $9.17/min	$150/18 min = $8.33/min
Demand in units	100	100	100
Units to manufacture	0	100	67
Throughput/manufactured unit	$90	$110	$150
Units contracted outside	100	0	33
Throughput/contracted unit	$64	$82	$92
Total product throughput	$6400	$11 000	$13 067
Total facility throughput		$30 467	
Operating expenses		$12 000	
Net profit		$18 467	

Table 3. Standard TOC analysis.

manufacture 100 units—the full market demand. Product Q with a throughput per constraint minute ratio of \$8.33/constraint minute, is next. Due to resource B's constrained capacity we only manufacture 67 units and outsource 33 units to satisfy the full market demand. Product P's ratio is the lowest (\$7.50/constraint minute), and no more manufacturing capacity is left available. We outsource the full 100 unit market demand for product P.

By computing the throughput of this solution (table 3) in the same method as above, we obtain a net profit of \$18 467 per week.

The TOC solution exceeds standard accounting throughput by an additional \$1267, which is a 7% increase over the standard accounting throughput.

The next section presents the LP analysis as applied to TOC principles.

7. LP analysis

The LP analysis presented above models the outsourcing problem as:

$$\text{Max} \sum_{i=1}^{M} X_i(CP_i - RM_i)(*)$$

Subject to:

$$\sum_{i=1}^{M} a_{ij}X_i \leq b_j; \quad j \in \{1, \ldots, N\} \text{ Manufacturing resource constraints.}$$

$$X_i \leq DQ_i; \quad i \in \{1, \ldots, M\} \text{ Product demand constraints.}$$

When applied to our example, this formulation writes:

$$\begin{aligned} &\text{Max}(66-40)X_p + (68-40)X_q + (98-40)X_r \\ \text{s.t.} \quad & 2X_p + 4X_q + 13X_r \leq 2400 \\ & 14X_p + 12X_q + 18X_r \leq 2400 \\ & 4X_p + 10X_q + 10X_r \leq 2400 \\ & 4X_p + 6X_q + 10X_r \leq 2400 \end{aligned}$$

The solution to this problem (as produced by LINDO) is: $X_p = 0; X_q = 50$ and $-X_r = 100$.

The same solution is reached via the modified Contractor Markup-to-Constraint minutes ratio (table 4). Contractor markup is computed by subtracting the cost of raw materials from the price charged by the contractor for one unit of final product. This markup is then divided by the time in minutes to compute the contractor markup per constraint minute. According to the LP criterion, product R with the highest ratio (\$3.22/constraint minute) is the first to be considered for production, followed by product Q with a lower ratio of \$2.33/constraint minute. The capacity of resource B allows production of only 50 product Q units. The remaining 50 units to satisfy the market demand are outsourced as are 100 units of product P—the product with the lowest markup per constraint minute (\$2.17/constraint minute).

This solution is identical to the LP solution.

The throughput of the LP solution is computed in an identical way to the preceding models. The \$19 000 throughput of this solution is \$1800 (or 10%) higher

	Product P	Product Q	Product R
Contractor price	$66	$68	$98
Raw material cost per unit	$40	$40	$40
Contractor markup	$26	$28	$58
Constraint resource B min/unit	12	12	18
Contractor markup per constraint minutes	$26/12 min = $2.17/min	$28/12 min = $2.33/min	$58/18 min = $3.22/min
Demand in units	100	100	100
Units to manufacture	0	50	100
Throughput/manufactured unit	$90	$110	$150
Units contracted outside	100	50	0
Throughput/ contracted unit	$64	$82	$92
Total product throughput	$6400	$9600	$15 000
Total facility throughput		$31 000	
Operating expenses		$12 000	
Net profit		$19 000	

Table 4. LP analysis of the TOC model.

	Standard accounting	TOC	LP
P Manufactured quantity:	100	0	0
Q	100	100	50
R	0	67	100
P Outsourced quantity:	0	100	100
Q	0	0	50
R	100	33	0
Net Profit	$17 200	$18 467	$19 000

Table 5. Summary of the three methodologies.

than the throughput of the standard accounting solution, and $533 (or 3%) higher than the throughput of the standard TOC solution.

Table 5 summarizes the differences between the three methodologies in terms of Manufacturing and outsourcing quantities and net profit.

The above example was constructed so that product P was the most preferred for in-house manufacturing for the standard accounting solution, product Q was most preferred for in-house manufacturing for the standard TOC solution and product R was most preferred for in-house manufacturing for the LP enhanced solution.

8. Conclusion and implications

This paper started with the motivation for the outsourcing problem, produced a formal definition of it, formulated it as a linear programming problem, developed the objective function and offered a simplified criterion for the ordering products in terms of preference to manufacture versus preference to outsource. This model was demonstrated to be analytically robust and, at the same time, simpler to implement. Thus the LP generated ratio requires only two variables per product: contractor markup and work time at the bottleneck.

The standard accounting solution is inferior to the TOC solution due to its treatment of all resources as equal. Such an assumption is unrealistic and only applies when all resources have identical utilization ratios. The TOC solution is inferior to the LP enhanced solution since it computes the throughput relative to a no production alternative while the LP solution computes the throughput based on the contractor's markup.

The LP solution has game theoretic implications for the whole market. It guides each player in the market to manufacture those products where the market is least efficient in terms of markup per constraint minutes and to purchase those products where the market is most efficient. This approach can be viewed as an application of TOC principles to the market as a whole. In a global market, different companies have bottlenecks at different resources. It is most probable that the contractor can offer the lowest markup for a certain product due to the fact that the resource that constrains the customer is not a constraint (a free resource) for the contractor. This enables the customer to elevate the system's constraints (the fourth step in TOC's five step methodology) while enabling the contractor to exploit non-constraint resources.

Finally, this tool is valuable to contractors competing on contracts. If costs are comparable among competing contractors, knowledge about the customer's constraint provides insight into the customer's value chain (Porter and Millard 1985) and decision-making process. A better-informed contractor can offer better terms per customer bottleneck time so as to improve the chance of more business.

References

COMAN, A. and RONEN, B., 1995, The enhanced make or buy decision: the fallacy of traditional cost accounting and the Theory Of Constraints. *Journal of Human Systems Management*, **14**(5), 303–308.

GOLDRATT, E. M., 1988, Computerized shop floor scheduling. *International Journal of Production Research*, **26**(3), 429–42; 1991, *The Haystack Syndrome* (Groton-on-Hudson, New York: North River Press).

FLOYD, B. and RONEN, B., 1989, Where Best to System Invest. *Datamation*, November 15.

FOX, R. E., 1988, *The Constraint Theory*. NAA Conference, Monville, NJ.

PORTER, M. E. and MILLARD, V. E., 1985, How information gives you competitive advantage. *Harvard Business Review*, July–August, 149–160.

RONEN, B. and STARR, M. K., 1990, Synchronized manufacturing as in OPT: from practice to theory. *Computers and Manufacturing*, August, 585–600.