

Optimal scheduling of purchasing orders for large projects

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Abstract: In this paper we generalize the newsboy model to optimize the scheduling of purchasing orders for large projects. The timely completion of a typical project hinges upon receiving all the purchased components by the time they are scheduled to be used. We show how to minimize the total expected cost of the project, including lateness penalty cost and holding costs. The main contribution of the paper is in solving the problem for more than one component while considering the interdependencies that exist among them. We outline an analytic solution and present fast heuristics.

Keywords: Project management; PERT; Purchasing; Newsboy problem; Lead time; BPERT

1. Introduction

In the project management environment we frequently face the need to schedule purchasing of project items (i.e., items that are not stocked regularly). For instance, take the following typical situation: a project manager has to purchase components for an assembly project. The project requires 1000 different purchased components, 600 of which are stocked regularly. The other 400 are specific to the project, and need to be purchased as part of the project execution. In addition, information about the historic lead time distribution for each item is available. If the items arrive before time, the project will be debited for holding costs at a rate of 28% per year (according to the internal cost accounting rate). If some items are late, the project will be delayed. Such delays bear a penalty of 2% of the total revenue per month. The items that arrive in time continue to accrue holding costs. In addition to the penalty, delays cause loss of goodwill, which is difficult to assess in monetary terms. (It might be \$20000 per month or \$200000.) The tradeoff between the costs of early and late arrival of parts creates an optimization problem.

Once we find the optimal scheduling of the purchasing orders, the expected costs can be calculated as a function of the due date and the lateness penalty. Therefore, this information is very useful while planning the bid. Thus, to assess the expected costs correctly the manager should plan the purchasing

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orders while still at the bidding stage, at least schematically. A model such as ours can be very useful for this purpose.

Before proceeding with our discussion, we present a brief survey of published results in two areas: (i) inventory theory, and (ii) stochastic project management. Delivery lags or lead times are discussed by Whybark and Williams (1976). They used simulation to show that time buffers are preferred over safety stock when the variation is in lead time (as opposed to demand fluctuations). For a more recent treatment of lead time problems see Karmarkar (1987).

The stochastic project management literature is mainly concerned with assessing the probability an activity will become critical. This is done under an implicit assumption that activities should start as early as possible ('early start'). For example see Dodin and Elmaghraby (1985). One notable exception is Britney (1976). Britney's model is concerned with estimation errors in activities durations, when two different linear penalties are assigned to under- and overestimating. Britney suggests the use of a Bayesian PERT model (BPERT) where the planned duration of each activity is calculated to minimize the penalties. He then shows that the best estimates are such that the probability of underestimation is equal to the ratio between the overestimation penalty and the sum of both penalties. This is similar to the results of the newsboy model. Britney applies his model for each activity separately. Looking at the whole project, he outlines a heuristic iterative procedure that attempts to assign under- and overestimation penalties to each activity based on the project's earliness bonus and lateness penalty. His analytic results are limited to single activities, and do not consider interdependencies. A similar application of the newsboy model to lead-time planning of a single item is given by Ballou (1985, p. 479).

This paper extends the newsboy model approach to a set of n items that arrive independently. The contributions include

- (i) an analytical model that can be solved to optimality by numeric methods;
- (ii) lower and upper bounds on the optimal order-placement time; and
- (iii) an efficient heuristic solution.

Section 2 introduces the problem by a basic case, where two items are involved. In section 3 we present a set of equations, which solve the general n -item case. This solution requires a tedious numerical search procedure. To facilitate the solution, and to approximate it, we introduce bounds on the ordering times and a fast heuristic. In Section 4 we tackle the issue of incorporating our model into the PERT/CPM environment. This includes a mathematical adaptation, and qualitative discussion. Section 5 shows some simple examples of using the techniques introduced in the paper. Finally in Section 6, we discuss some other applications, and suggestions for future extensions.

2. Purchasing decisions for one or two items

2.1. The one-item case

A project requires one purchased component, which must be on hand at a specified time, t^* . The lead time of the component has a given stationary stochastic distribution. If the item is received earlier than t^* , say at time t , then a holding cost of $C(t^* - t)$ will be incurred. If the component is late, the whole project is delayed. Therefore, the project lateness penalty, P , is incurred for each time unit of delay. The objective function is:

$$\min_T \{ E(\text{Penalty cost}) + E(\text{Holding cost}) \} \quad (2.1)$$

where T is the order placement time. Figure 1 illustrates the relationship between t^* , T and the lead time distribution.

Expanding the objective function (2.1), we may write:

$$\min_T \left\{ C \int_T^{t^*} F(t - T) dt + P \int_{t^*}^{\infty} [1 - F(t - T)] dt \right\} \quad (2.2)$$

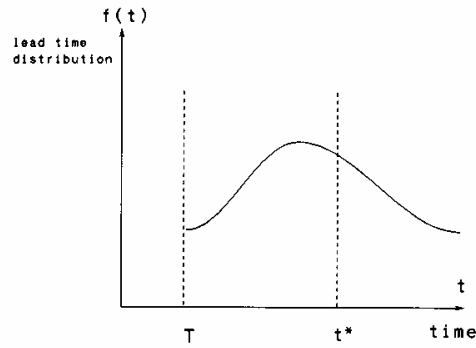


Figure 1. The one-item case: the relationship between the specified due data (t^*), the order replacement time (T) and the lead time distribution ($f(t)$)

where

$F(\cdot)$ = CDF of the lead time.

C = Holding cost per period.

P = Penalty cost per period.

If T is unconstrained, the result is simply

$$F(t^* - T^*) = \frac{P}{P + C}. \tag{2.3}$$

(See Ballou, 1985, for the derivation.) T cannot be negative, however, and therefore the formal solution is

$$T^* = \max\left\{t^* - F^{-1}\left(\frac{1}{1 + C/P}\right), 0\right\}. \tag{2.4}$$

2.2. The two-item case

Now assume that the project requires two independent items. It is enough that one item will be delayed, to delay the whole project, and thus incur the penalty cost P . If one of the items arrives in time, however, and the other is delayed, we also have to carry the holding cost for the item that arrived.

For simplicity, we assume that the items are required at the same time t^* . In Section 4 we drop this assumption. Let C_i denote the inventory holding costs per period of item i ($i = 1, 2$). Let $F_i = F_i(t - T_i)$ denote the probability that item i will arrive before t , given that it was ordered at T_i . We assume that the F_i are independent of each other. Let P be the project delay penalty cost per period.

For convenience, we calculate the expected costs until t^* and from t^* onwards separately. The expected holding cost until t^* , $h = h(T_1, T_2)$, is

$$h(T_1, T_2) = C_1 \int_{T_1}^{t^*} F_1(t - T_1) dt + C_2 \int_{T_2}^{t^*} F_2(t - T_2) dt. \tag{2.5}$$

From t^* onwards, at any given time t , there are four mutually exclusive and exhaustive possible random events:

1. Both items are on hand, and the implied cost is zero.
2. Item 1 is on hand, but item 2 is missing, resulting in a cost of $P + C_1$.
3. Similarly, where item 1 is missing, the cost is $P + C_2$.
4. Both items are missing at the same time, with cost P .

Combining these costs with h , we obtain the target function as follows:

$$Z = Z(T_1, T_2) = h + (P + C_1) \int_{t^*}^{\infty} F_1(1 - F_2) dt + (P + C_2) \int_{t^*}^{\infty} (1 - F_1)F_2 dt + P \int_{t^*}^{\infty} (1 - F_1)(1 - F_2) dt. \quad (2.6)$$

Our objective is to minimize Z , by optimizing the order points, T_1^* and T_2^* . To that end, we take the partial derivatives (using the Leibnitz method where required), and set them to zero. This yields the following equation:

$$C_1 = (P + C_1 + C_2) \int_{t^*}^{\infty} \frac{\delta F_1}{\delta(t - T_1^*)} F_2(t - T_2) dt. \quad (2.7)$$

A symmetrical result holds for the second time. These equations can be solved numerically.

3. Purchasing decisions for n independent items

3.1. The n -item case

In this section we continue to assume that all the items are needed at the same time, t^* . Let $h(T)$ (where $T = (T_1, T_2, \dots, T_n)$) denote the expected holding costs until t^* ; then

$$h(T) = \sum_{i=1}^n C_i \int_{T_i}^{t^*} F_i(t - T_i) dt. \quad (3.1)$$

From t^* onwards, at any given time t , there are now 2^n mutually exclusive and exhaustive possible random events, which can be categorized as follows:

1. All items are on hand, and the implied cost is zero.
2. All items except item i are on hand, yielding a cost of $P + \sum_j C_j - C_i$.
3. Item i is on hand, but at least one other item is missing, resulting in a cost of $P + C_i$ + possibly some other C_j 's.
4. Similarly, where item i is missing and at least one other item is missing, the cost is P + possibly some other C_j 's.

Combining these costs with h , we obtain the objective function. For convenience we start with T_1 , and generalize for T_i later by symmetry. Note that case 1 can be paired with case 2, and all the cases of 3 form 2^{n-1} similar pairs with cases of 4. Equation (3.2) is arranged according to these pairs. For clarity we also show the entry for case 1, where all items arrived, even though its contribution is zero.

$$Z = Z(T) = \sum_i C_i \int_{T_i}^{t^*} F_i dt + [\text{holding costs till due date}] \quad (3.2)$$

$$\left. \begin{aligned} & 0 \int_{t^*}^{\infty} F_1 F_2 \cdots F_n dt + [\text{all items on hand: case 1}] \\ & \left[P + \sum_{j \neq 1} C_j \right] \int_{t^*}^{\infty} (1 - F_1) F_2 \cdots F_n dt + [\text{item 1 missing: case 2}] \end{aligned} \right\} \text{pair 1}$$

$$\left. \begin{aligned}
 & \left[P + C_1 + \sum_{j \neq 1,2} C_j \right] \int_{t^*}^{\infty} F_1(1 - F_2)F_3 \cdots F_n dt + \\
 & \quad \text{[item 1 on hand but item 2 missing: case 3]} \\
 & \left[P + \sum_{j \neq 1,2} C_j \right] \int_{t^*}^{\infty} (1 - F_1)(1 - F_2)F_3 \cdots F_n dt + \\
 & \quad \text{[items 1 and 2 missing: case 4]} \\
 & \vdots \\
 & (P + C_1) \int_{t^*}^{\infty} F_1(1 - F_2) \cdots (1 - F_n) dt + \text{[items 2 - } n \text{ missing: case 3]} \\
 & P \int_{t^*}^{\infty} (1 - F_1)(1 - F_2) \cdots (1 - F_n) dt \text{ [all items missing: case 4]}
 \end{aligned} \right\} \begin{array}{l} \text{pair 2} \\ \\ \\ \\ \\ \text{pair 2}'' \end{array}$$

To minimize Z , we have to set its partial derivatives with respect to T_i to 0. We continue to concentrate on the partial derivative with respect to T_1 . For this purpose, only some of the elements of (3.2) are pertinent, the others not being functions of T_1 . Looking at h and the first pair, their applicable part is

$$C_1 \int_{T_1}^{t^*} F_1 dt - \left[P + \sum_{j \neq 1} C_j \right] \int_{t^*}^{\infty} F_1 \prod_{j \neq 1} F_j dt. \tag{3.3}$$

Each of the other 2^{n-1} pairs contributes

$$C_1 \int_{t^*}^{\infty} F_1 \prod_{j \neq 1} [\text{some combination of } (1 - F_j) \text{ or } F_j] dt. \tag{3.4}$$

If we sum all the values for the various combinations of $1 - F_j$ or F_j in (3.4), all the possible combinations, except the case where all the items arrived, are represented. Therefore the sum is:

$$C_1 \int_{t^*}^{\infty} F_1 \left(1 - \prod_{j \neq 1} F_j \right) dt \tag{3.5}$$

By regrouping and some algebraic manipulations we finally get the following expressions:

$$\frac{\delta Z}{\delta T_1} = \frac{\delta}{\delta T_1} \left[C_1 \int_{T_1}^{\infty} F_1 dt - S \int_{t^*}^{\infty} \prod_j F_j dt \right] \tag{3.6}$$

where $S = P + \sum_i C_i$. Noting that $(\delta[\int_{T_1}^{\infty} F_1 dt]/\delta T_1) = 1$, and setting (3.6) to zero, we obtain

$$C_1 = S \int_{t^*}^{\infty} \frac{\delta F_1}{\delta(t - T_1)} \prod_{j \neq 1} F_j(t - T_j) dt \tag{3.7}$$

where, for convenience, we treat $t - T_1$ as a variable. This is permissible since T_1 is held constant during the differentiation. By replacing the index 1 in (3.7) by i , $i = 2, 3, \dots, n$, we obtain a set of nonlinear equations, which can be solved numerically. For that purpose, any nonlinear search method may serve. In particular, if F_j is differentiable, a quasi-Newton search may be used for that purpose. Note that (2.7) is a special case of (3.7), as expected.

3.2. Evaluating upper and lower bounds

We now evaluate upper and lower bounds (designated F^+ and F^- , respectively) for the T_i 's. Since $\prod_j F_j < 1$,

$$\frac{C_i}{S} < \int_{t^*}^{\infty} \frac{\delta F_i}{\delta(t - T_i)} dt. \quad (3.8)$$

But the right-hand side of (3.8) is the probability item i will be late. Rewriting (3.8) in terms of the probability item i being on time we obtain the upper bound F_i^+ on its optimal value, F_i^* :

$$F_i^* \leq \int_{t^*}^{\infty} \frac{\delta F_i}{\delta(t - T_i)} \prod_{j \neq i} F_j dt < 1 - \frac{C_i}{S} = F_i^+. \quad (3.9)$$

Note that the upper bound on F_i can be translated directly to a lower bound on the ordering time. This bound can serve as an approximation, or as a first guess for the exact solution. The bound is especially close if the penalty, P , is very high relative to *all* holding costs. Using this bound as an approximate solution increases the holding costs and decreases the penalty cost relative to the optimal solution. An important advantage of this approximation is that it decomposes the n -item problem to n instances of the simple single-item model. Thus it is very easy to calculate, even for large projects.

We can also look for lower bounds for the F_i^* -values. If we use such lower bounds to approximate the solution, the penalty may be too high and the holding costs will be kept down. As it turns out, in some cases such lower bounds can be obtained. Denote the lower bound we look for by F_{-i} ; then by observation of (3.7), we obtain

$$F_i^* > 1 - \frac{C_i}{S \prod_{j \neq i} F_j^*} > 1 - \frac{C_i}{S \prod_{j \neq i} F_{-j}}. \quad (3.10)$$

Therefore, if we can solve the following set of equations, subject to all values of F_{-i} being feasible (i.e., $0 < F_{-i} \leq F_i(t^*) < 1$), we obtain a set of lower bounds, as required:

$$F_{-i} = 1 - \frac{C_i}{S \prod_{j \neq i} F_{-j}}. \quad (3.11)$$

We now show that (3.11) can be solved by a search over a single variable. First, for convenience, we define $A_i = C_i/S$. Next, note that $F_{-i} \neq 0$, so we can multiply and divide the right-hand side of (3.11) by F_{-i} , to obtain:

$$F_{-i} = 1 - \frac{A_i F_{-i}}{\prod_j F_{-j}}. \quad (3.12)$$

Letting $x = \prod F_{-j}$ and solving for F_{-i} , we get

$$F_{-i} = \frac{x}{x + A_i}. \quad (3.13)$$

By the definition of x and (3.13), we must have

$$x = \prod_i \left[\frac{x}{x + A_i} \right]. \quad (3.14)$$

Finally, since $0 < x < 1$, it is easy to verify that (3.14) is satisfied if and only if

$$g(x) = \prod (x + A_i) - x^{n-1} = 0, \quad (3.15)$$

for some x in this domain.

Thus we have reduced our problem to a search over a single variable, x . Once x is found, (3.13) is used to obtain F_{-j} . Since such searches are easy to perform, we can practically say that the procedure is $O(n)$. By specifying large enough values for C_i , however, we can construct examples where (3.15) does not have a solution in the specified domain. For instance, try $C_1 = C_2 = P$, where (3.15) does not have a solution at all, let alone one in the specified domain.

To complete the discussion about the calculation of the lower bound, F_{-i} , we consider the case when some of the items have to be ordered immediately. Let J be the set of items ordered immediately, and hence their respective F_{-j} are predetermined and do not satisfy (3.13). By observation of (3.11), if for the other items we divide the A_i -values by $\prod_{j \in J} F_{-j}$, we get the same model as before. The same observation indicates that if $\prod_{j \in J} F_{-j}$ is low, then the A_i -values become relatively large. This makes it more likely that the bound will not exist.

We now present a new heuristic, which we call *The Iterative Heuristic*. The Iterative Heuristic avoids the pitfall of ordering too soon when the probability other items will arrive in time is low. In contrast, when we use the upper bound procedure, even if one of the items is not likely to arrive in time, all the other items are ordered early. As a result we have to pay the delay penalty *and* high holding costs. A more rational behavior is to order the other items later, and save on the holding costs. The idea of the heuristic is to solve for the F_i -values together, starting with values derived by applying the single item model to each item separately, and to adjust them iteratively. The major difference between The Iterative Heuristic and the upper bound procedure is that now we use a different penalty for each item instead of S . This penalty is based on the probability that the other items arrived.

The penalty we propose is:

$$P_i = \frac{P + \sum_{j \neq i} F_j C_j}{n - \sum_{j \neq i} F_j} \quad \text{for all } i. \quad (3.16)$$

The assumption behind this penalty is that if item i did not arrive, then the expected number of items that did not arrive is $n - \sum_{j \neq i} F_j$ (note that this is always > 1), and all these share the penalty equally. The expected penalty is $P + \sum_{j \neq i} F_j C_j$.

Given (3.16) as our penalty for item i , F_i should be

$$F_i = \frac{P_i}{C_i + P_i} \quad \text{for all } i. \quad (3.17)$$

We need to apply (3.16) and (3.17) iteratively until they converge to yield stable F_i - and P_i -values. The following lemma gives the solution for cases where all the probabilities are unconstrained – i.e., as per (3.17).

Lemma. Let T_i satisfy $F_i(t^* - T_i) = P/(P + C_i)$. Then if $T_i > 0, \forall i$, we have $P_i = P, \forall i$.

Proof. We have to show that setting $P_i = P$ in (3.16) and (3.17) will yield a stable solution. That is, we have to show that

$$P = \frac{P + \sum_{j \neq i} P C_j / (P + C_j)}{n - \sum_{j \neq i} P / (P + C_j)}.$$

Dividing both sides by P and rearranging, we have to show that

$$1 + \frac{\sum_{j \neq i} C_j}{P + C_j} = n - \frac{\sum_{j \neq i} P}{P + C_j}.$$

Substituting $n = 1 + \sum_{j \neq i} 1$ we obtain

$$1 + \frac{\sum_{j \neq i} C_j}{P + C_j} = 1 + \sum_{j \neq i} \left(1 - \frac{P}{P + C_j} \right).$$

It is easy to see that the last equality is true. \square

Based on the Lemma, we recommend starting The Iterative Heuristic with $P_i = P$. If all the ordering times that result are unconstrained, we do not have to iterate at all.

We now show that The Iterative Heuristic converges to a stable solution. The line of argument is:

- (i) If we start running the procedure setting $P_i = P$, P_i will be monotone non-increasing, and
- (ii) P_i is bounded from below, so there is a limit as to how much it can decrease.

In more detail,

(i) Assume F_i is indeed monotone non-increasing; then the denominator of (3.16) is monotone non-increasing, and the numerator is monotone non-decreasing, which implies (3.16) itself is monotone non-increasing from iteration to iteration.

- (ii) $P_i \geq P/n$, with equality if and only if $F_j = 0, \forall j$.

4. Adjusting the model to data generated by PERT

So far we have assumed that the project due data, t^* , is also the item due date for all the items. In reality, however, the items are likely to be required before the project due date. Furthermore, each of them may be required at a different time. In this section we show how to adapt the model to this reality.

Our policy is to minimize the WIP (work-in-process). Therefore we assume that activities start at their respective latest start time. The environment of assembly projects is such that the variance of the assembly durations (once the components have arrived) is relatively low. Hence we may be justified in using a deterministic CPM network analysis.

Alternatively, when the activities are stochastic, we can adopt Britney's BPERT durations (Britney, 1976). As proposed by Britney, these Bayesian estimates are treated exactly as if they were deterministic. To do this, Britney's model requires over- and underestimation penalties. For the overestimation penalty we propose to use $\sum C_i$, where the summation is over all the items that are scheduled to have arrived by the time the activity starts. For the underestimation penalty we propose using $\sum P_i$, where P_i is as per The Iterative Heuristic and the summation is over all the items used in the activity itself. (In case we overestimate the duration of an activity, it will be finished ahead of time, and we will have to pay holding costs. If we underestimate, the activity will be late and we will have to pay a delay penalty.)

After doing this, we obtain for each item a separate due date, corresponding to the latest start of the activity that requires it. Suppose now that the latest start time for an item is three months before the project due date. Conceptually, all we have to do is to place the order for that item three months earlier than the time computed for it before. Again, if this adjustment causes the ordering time to be negative, we order immediately, and adjust F_i accordingly. This can be achieved in a more formal manner as follows:

- Let t_i^* ($i = 1, \dots, n$) be the time item i is required.
- Let $\Delta t_i = t^* - t_i^*$ (or: $t_i^* = t^* - \Delta t_i, \forall i$).
- Let $t_i = t - \Delta t_i$ (or $t = t_i - \Delta t_i$, any i).

- Let T_i^* be the optimal time to order item i .
- Let $F_i^* = F_i(t_i^* - T_i^*)$.

Then instead of (3.7) we now use:

$$C_i = S \int_{t_i^*}^{\infty} \frac{\delta F_i}{\delta(t_i - T_i^*)} \prod_{i \neq j} F_j(t_j - T_j^*) dt, \quad i = 1, \dots, n. \quad (4.1)$$

Similarly, instead of (3.9), we now have:

$$T_{-i} = \max \left\{ \arg \left\{ F_i(t_i^* - T_{-i}) = 1 - \frac{C_i}{S} \right\}, 0 \right\} \quad (4.2)$$

where T_{-i} is a lower bound on T_i^* which corresponds to the upper bound on F_i^* . Using this approach we can also adapt the lower bound search procedure (i.e., (3.11)) – which will yield an upper bound on the ordering time – and The Iterative Heuristic. We omit the details.

Though it is not our purpose in this paper to concentrate on sequencing the PERT activities, we should note here that if the given sequence yields negative ordering times (i.e., immediate orders with lower than optimal on-time-arrival probabilities), other sequences may have to be considered that may be less problematic in this sense. One way to incorporate this type of consideration into the sequencing algorithm, may be to impose early start constraints based on the pertinent items' lead time distributions. Such constraints, however, need to be 'soft' in case they lead to infeasible schedules.

5. Examples

Let us first assume that a project requires only one component. Consider the following special case: We have to use a certain item eight months from now. The item's lead time distribution is exponential with an expected value (or a parameter μ) of 2 months. The carrying cost of this item is 18% per year on a cost of \$1200, and the penalty cost is \$600 per month. Thus, $t^* = 8$ months, $\mu = 2$ months, $C = \$18$ per month, $P = \$600$ per month.

Our lead time distribution is exponential, i.e., $F(t) = 1 - \exp(-t/\mu)$. Using the solution of (2.4) leads to

$$T^* = t^* + \mu \ln \left(\frac{C}{C + P} \right) \quad (5.1)$$

and we get $T^* = 0.927$ months.

Let us now assume that we have 20 components with the same behavior. First, we use the upper bound on F_i to obtain a lower bound on the ordering time, T_{-i} . That is, we specify $F_i(t_i^* - T_{-i}) = 1 - C_i/S$, as per (3.9). In our case $S = 960$, while $C_i = 18$ as before. Thus,

$$T_{-i} = t_i^* + \mu \ln \left(\frac{C_i}{S} \right), \quad (5.2)$$

which leads to $T_{-i} = 0.047$ months (i.e., tomorrow afternoon).

The probability the project will be on time in the single-item example is $600/618 = 0.97$. In the 20-item case this probability is reduced to $(960/978)^{20} = 0.69$, even though we order earlier. This result, while intuitively clear, indicates the importance of good planning for multi-item projects. Note also that if we specify more than 22 similar items, the bound will yield a negative ordering time, i.e., immediate orders.

As for the lower bound, F_{-i} , $x = 0.4$ solves (3.15) in this case, yielding $F_{-i} = 0.955$ ($i = 1, 2, \dots, 20$), which implies an upper bound on the ordering time of 1.79 months.

Next, let us apply The Iterative Heuristic to the same problem, with 20 items. As discussed above, as long as the probabilities are unconstrained, this implies using the single-item model for each item separately. Thus, we are led to order after 0.927 months as in the single-item case. This is because without binding constraints the last heuristic assigns a penalty of P to each item separately. The heuristic calls for ordering almost exactly midway between the bounds in this case (0.047 and 1.79 months). Note that in this case the lower bound on T is unconstrained; therefore, the optimal solution must also be unconstrained.

The real optimal solution for this example is $T_i^* = 0.475$ months, which happens to be halfway between the lower bound and the heuristic. To gain more insight into the behavior of the problem, we calculated the objective function value at the various solutions for comparison. The optimal solution's value is 2791 (2006 holding cost, and 785 penalty). The lower bound T_{-i} (0.047) yields 2804, and the heuristic 2806 – both within 0.54% of the minimum. The upper bound (1.79) yields 2928 – about 4.9% above the minimum. As might be expected, the function is quite flat around the minimum.

To make the problem more challenging, we now assume that five of the items are required in four months, and their lead time average $\mu = 4$ months. The other 15 items are needed after eight months and with $\mu = 2$ months, exactly as before.

The upper bound on F_i^* remains intact. Because of the different due date and μ -value for the five items, however, they should be ordered immediately. Even if we do that, the probability that they will arrive on time is only 0.632 for each of them, as opposed to 0.982 called for by the bound. The other 15 items are ordered after 0.047 months, as before.

For the lower bound, as discussed above, we adjust A_i for the 15 items by dividing them by 0.632⁵, thus inflating them almost tenfold. As a result, (3.15) does not have a proper solution any more.

This leaves us with The Iterative Heuristic as the only easy method to assess the impact of the constrained items on the others. Applying (3.16) and (3.17) iteratively now yields $P_i = 252$ for the five items (which still implies they should have been ordered 6.8 months ago), and $P_i = 231$ for the other items. The result is that we order the 15 items after 2.65 months.

The optimal solution in this case is to order after 3.44 months. Since $T_{-i} = 0.047 < 2.65 < 3.44$, we can conclude that in this case the heuristic outperforms the bound significantly. Table 1 summarizes the example's results.

Observing the behavior of The Iterative Heuristic, in the first case, where the probabilities are unconstrained, it yields the result of postponing *all* orders relative to the upper bound on F . This is surprising because it postpones the expected completion date, which is quite likely to be late. The optimal solution, however, also specifies postponing all orders, though by less in this case. In the second case, where some of the probabilities are constrained, it makes sense to save on the holding costs of the other items, and the heuristic does so effectively. In conclusion it seems that the heuristic is the best choice in the presence of constrained probabilities.

Table 1
Summary of the example

	Order time (months)				Probability of on-time completion			
	T^*	T_-	T^+	$T_{\text{heur.}}$	Optimal	with T	with T^+	with heur.
(a) Order time and probabilities								
Single item	0.927	0.927	0.927	0.927	0.971	0.971	0.971	0.971
20 items	0.475	0.047	1.97	0.927	0.625	0.685	0.400	0.554
5/15 items	0/3.44	0/0.047	0/NA	0/2.65	0.020	0.076	NA	0.035
(b) Sensitivity analysis								
	Objective function value							
	Z^*	Z_{T_-}	Z_{T^+}	$Z_{\text{heur.}}$				
20 items	2791	2804	2928	2806				

Our example indicates that when the probabilities are not constrained, both the bound T_i and the heuristic are viable methods. Other numerical examples, however, indicate that the heuristic is preferable in case the holding costs are high relative to the penalty. In such cases the bound tends to be farther from the optimal solution.

6. Conclusion

We have developed an analytic solution to the multi-item time-based newsboy problem. This requires numeric search methods. To facilitate the search, and to provide approximate solutions, we developed bounds on the optimal ordering time, and a heuristic solution. Hi-tech projects, and modern production in general, are characterized by an ever growing need for timeliness. Management, customers and competitors, push for shorter cycle times. Thus a model such as ours may fill an important gap. Decision Support Systems, such as the one proposed by Ronen and Trietsch (1988), may be very useful to practitioners.

An interesting research area is the dynamic problem, where we wish to update the schedule as the project progresses. For instance, suppose we already know that a certain problematic item has arrived. Then we may wish to order other items sooner than originally planned. In some cases it may be possible to stipulate in advance that certain orders should be postponed until some time after the arrival of other (long lead time) items.

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