An information-economics approach to quality control attribute sampling

Boaz Ronen

Faculty of Management, The Leon Recanati Graduate School of Business Administration, Tel Aviv University, Tel Aviv 69 978, Israel

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Abstract: The information-structures model provides a theoretical framework for assessing the normative value of information. This paper applies the information-structures model to quality control attribute sampling, which is usually based on industrial and military standards that do not involve economic considerations or meet cost considerations. Common quality control terms, such as AQL, LTPD, and operating characteristic curves, are shown as special cases of the information-structures model. Theorems involving dominance among various quality control plans are proved. The Blackwell Theorem on the 'generally more informative' relationship is modified for this application.

Keywords: Information economics; Information Structures; Value of information; Informativeness; Quality control

1. Introduction

The information-economics approach to evaluating the value of information is based on the information-structures (IS) model developed by Marschak (1971) and McGuire and Radner (1986) and later expanded by Demski (1972), Ahituv (1981), Ahituv and Ronen (1988), Ahituv and Wand (1984)), and others. The model describes the information system as a stochastic (Markov) matrix of probabilities which transforms states of nature to signals. The decision maker must select the optimal action under given values of a priori probabilities of states of nature, and given values of payoffs. The information-economics model proposes a partial rank ordering of information structures by using Blackwell's Theorem (McGuire and Radner, 1986).

There are but few applications of information economics in general and of the information-structures model in particular. Stohr (1979) uses the information economics approach for optimal observing of inventory levels. Examples of quality control problems using the information-structures model were demonstrated by Wallock and Adams (1963) and Demski (1972). Further research conducted by Moskowitz and Berry (1976) suggested a method for finding an optimal sample. This paper's contribution is twofold: first, it applies the information-structures theory and methodology to the practical area of

Correspondence to: Dr. B. Ronen, Faculty of Management, Tel Aviv University, University Campus, Ramat Aviv, Tel Aviv 69 978, Israel.

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quality control (QC) attribute sampling. Then, it adds to the theory of quality control sampling by proving theorems on the selection and comparison of QC plans.

Section 2 of the paper reviews the information-structures model. In Section 3 we show that QC sampling can be viewed as a special case of the information-structures model. Section 4 rank orders information structures of QC sample plans in a 'generally more informative' way. In Section 5 we discuss the special and useful case of 2×2 information-structures. Section 6 shows the relationship between the informativeness of a sample and the sample size. In Section 7 conclusions are drawn.

2. The information-structures model: A review

This section briefly reviews the information-structures model. For further details the reader is referred to McGuire and Radner (1986), or Ahituv (1981). Let E be a finite set of events (states) of nature $E = \{e_1, \ldots, e_{n_E}\}$. Let π be the vector of a priori probabilities associated with the events in E.

$$\pi^t = (\pi_1, \ldots, \pi_{n_E}), \qquad \sum \pi_i = 1, \quad \pi_i \geq 0, \quad i = 1, \ldots, n_E,$$

where t represents the transpose operation.

Let Z be a finite set of signals $Z = \{z_1, \ldots, z_{n_Z}\}$. An information structure or information matrix, Q, is an $n_E \times n_Z$ Markov (stochastic) matrix of the conditional probabilities in which signals of the set Z will be displayed at the occurrence of events in E. Thus, q_{ij} of Q is the probability that for a given event e_i, signal z_j will be displayed. (If Q contains only 1 or 0 elements, then it is a noiseless structure.)

Let A be a finite set of feasible actions to be taken by the decision maker: $A = \{a_1, \ldots, a_{n_A}\}$. A cardinal payoff function, U, is defined from $A \times E$ to the real numbers, R, associating payoffs to pairs of actions and events, $U: A \times E \to R$. The function U can be depicted by an $n_A \times n_E$ matrix, U, whose elements reflect the payoff gained under any combination of action a_i of A and event e_j of E.

The decision maker does not observe the events, but only the signals, and chooses actions according to these signals. The decision maker's strategy can be described by an $n_z \times n_A$ Markov matrix, D, which contains the probabilities of taking certain actions after being stimulated by certain signals.

Thus, d_{ij} of D is the probability that for a given signal z_i, action a_j will be taken. (If D contains only 1 or 0 elements, then D is a pure strategy).

Let π be a square matrix containing the elements of π in its main diagonal, and zeros elsewhere:

$$\pi = \begin{vmatrix} \pi_1 & & 0 \\ & \ddots & \\ 0 & & \pi_{n_E} \end{vmatrix}.$$

Then the expected payoff of the combination of an information structure, Q, a decision rule (strategy) D, a payoff matrix U, and a probabilities vector π will be $\mathrm{tr}(QDU\pi)$, where tr( ) represents the trace operator (McGuire and Radner, 1986). Optimization is reached by finding a Markov matrix D^* out of all possible $n_Z \times n_A$ Markov matrices to maximize tr( ).

Let us define

$$F(Q, U, \pi) = \max\{\mathrm{tr}(QDU\pi)\}.$$

Let us, further, define the relationship

$$Q_A \geq Q_B \quad (Q_B \text{ is not better than } Q_A \text{ regarding } U \text{ and } \pi) \text{ if}$$
$$F(Q_A, U, \pi) \geq F(Q_B, U, \pi).$$

Q_A is regarded as generally more informative than Q_B (denoted by $Q_A \geq Q_B$) if Q_B is not better than Q_A for all payoff matrices U and all probability vectors π.

3.2. AQL, LTPD and the information structure model

P_0 and P_1 can normally represent AQL and LTPD levels of quality, as defined below:

AQL (*Accepted Quality Level*) is the quality level of a 'good' lot. It is the percentage of defectives that can be considered satisfactory as a process average and represents a level of quality which the producer wants accepted with a high probability of acceptance. P_0 can be chosen as the desired AQL.

LTPD (*Lot Tolerance Percent Defective*) is the quality level of a 'bad' lot. It represents a level of quality which the consumer wants accepted with a low probability of acceptance. P_1 can be the desired LTPD.

The *information matrix of a QC plan* is an $n_E \times 2$ Markov matrix whose rows represent states of nature P_i, $i = 1, \ldots, n_E$, and the columns are the aggregated signals $y \leq c$ and $y > c$ (where c is the acceptance number, $0 \leq c \leq n$; n is the sample size).

The elements of an information matrix are

$$q_{i1} = \Pr[y \leq c/P = P_i], \qquad q_{i2} = \Pr[y > c/P = P_i], \quad i = 1, \ldots, n_E.$$

Example 2. We shall now show an information structure of a QC plan having 2 states of nature: AQL and LTPD.

$$Q = \begin{array}{r} \\ P_0 = \text{AQL} \\ P_1 = \text{LTPD} \end{array} \begin{array}{c} \begin{array}{cc} y \leq c & y > c \end{array} \\ \left| \begin{array}{cc} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{array} \right| \end{array}.$$

The matrix represents the relations between the states of nature and the signals; in this example, a lot having a percentage of defects less than or equal to AQL is considered a 'good' lot, and a lot having P greater than or equal to LTPD defectives is considered a 'bad' lot. This presentation enables us to elaborate on two more QC terms:

Producer's risk (α):

$$\alpha = \Pr(\text{reject a lot/the lot is 'good'}),$$

$$\alpha = \Pr(y > c/P = \text{AQL}),$$

where y is the number of defectives. This producer's risk may also be viewed as a type I sampling error.

Consumer's risk (β):

$$\beta = \Pr(\text{do not reject a lot/the lot is '}\textit{bad}\text{'}),$$

$$\beta = \Pr(y \leq c/P = \text{LTPD}).$$

Thus, the presentation in the above example enables us to present all the elements of a QC plan (AQL, LTPD, n, c, α, and β) in a single matrix in terms of the information-economics approach. Clearly, any QC plan can be presented as an information structure. The consumer risk is the type II sampling error.

3.3. The decision matrix of a sample

D is a decision matrix of an $n =$ item sample if it is an $(n + 1) \times 2$ Markov matrix that associates signals with decisions. The signals are the number of defectives ($y = 0$ through $y = n$) and the decisions are 'accept' or 'reject' a whole lot.

Example 3. Assume a 5-item sample; one of the feasible decision matrices is

$$D = \begin{array}{c|cc|} & \text{Accept} & \text{Reject} \\ y=0 & 1 & 0 \\ y=1 & 1 & 0 \\ y=2 & 1 & 0 \\ y=3 & 0 & 1 \\ y=4 & 0 & 1 \\ y=5 & 0 & 1 \end{array}.$$

The strategy presented in this matrix is:–"Take a 5-item sample, accept the lot if the number of defectives is two or less; otherwise,–reject it". This matrix is, in fact, equivalent to the plan (5, 2).

Decision matrix of a plan: D will be called a 'decision matrix of a QC plan' if it is a 2×2 Markov matrix that associates aggregated signals with decisions.

The signals are

$y \leq c$ and $y > c$,

the decisions are

'accept the whole lot' and 'reject the whole lot'.

Example 4. The matrix of Example 3, corresponding to the plan (5, 2), can also be presented as the following decision matrix:

$$D = \begin{array}{c|cc|} & \text{'Accept'} & \text{'Reject'} \\ y \leq c & 1 & 0 \\ y > c & 0 & 1 \end{array}.$$

Note that some strategies do not need to be considered in the first place: for instance, strategies that always accept or reject the lot. Clearly, this case would not require an information system (a sample), since the decision is never affected by any of the signals.

Strategies for which $d_{ij} = \{0, 1\}$ are labelled 'pure strategies'. When at least one of the elements is neither one nor zero, the strategy is called a 'mixed strategy'.

3.4 The prior probability vector and the payoff matrix

The *prior probability vector* is a vector π,

$$\pi = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_{n_E} \end{pmatrix},$$

whose elements are the prior probabilities of the n_E states of nature, $P_1, \ldots, P_{n_E}$, (where $P_1 < P_2 < \cdots < P_{n_E}$). Note: π is defined for both matrices – plan and sample.

The *payoff matrix* U is a $2 \times n_E$ matrix in which each element u_{ki} displays the payoff related to a decision k ('accept' or 'reject') and to the occurrence of state of nature i.

Note: The payoffs can be expressed in terms of costs, which are estimated by the decision maker or derived from historical data stored in the information system of the organization. Some corporations which have adopted the 'quality costs' concepts can construct the payoff matrix almost instantly by extracting data from their data base.

Example 5. Suppose a 'good' lot is considered a lot having 2% defectives, and a 'bad' lot is one having 5% defectives. Suppose the a priori probabilities of these states of nature are $\pi^t = (0.9, 0.1)$.

The payoff matrix is

$$U = \begin{array}{r} \\ \text{Accept lot} \\ \text{Reject lot} \end{array} \begin{array}{cc} \text{'good' lot} & \text{'bad' lot} \\ \left|\begin{array}{cc} 0 & -1000 \\ -100 & 0 \end{array}\right| \end{array}.$$

Two QC plans are considered: $A = (158, 4)$ and $B = (184, 5)$. Assuming that the testing costs are equal for the two plans, which one is to be preferred? Using the terminology defined above, the solution may be drawn as follows:

Let Q_A and Q_B be the information matrices for plans A and B, respectively:

$$Q_A = \begin{array}{r} \\ P=0.02 \\ P=0.05 \end{array} \begin{array}{cc} y \le 4 & y > 4 \\ \left|\begin{array}{cc} 0.79 & 0.21 \\ 0.1 & 0.9 \end{array}\right| \end{array}, \qquad Q_B = \begin{array}{r} \\ P=0.02 \\ P=0.05 \end{array} \begin{array}{cc} y \le 5 & y > 5 \\ \left|\begin{array}{cc} 0.83 & 0.17 \\ 0.1 & 0.9 \end{array}\right| \end{array}.$$

The optimal decision rule for Q_A and Q_B turns out to be

$$D = \left|\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right|$$

and the expected payoffs are

$$\max_D \{\mathrm{tr}(RDU\pi)\} = -28.9 \text{ and } \max\{\mathrm{tr}(QDU\pi)\} = -25.3.$$

It may be seen that under these particular circumstances plan B (represented by Q) is more informative than A, and, therefore, preferred. If plan A is in use, then the switch over to plan B improves the performance by $-25.3 - (-28.9) = 3.6$.

We have defined two different information structures: an information structure of a sample (a sample matrix) and an information structure of a plan (a plan matrix). We shall now show and prove the relationship between these matrices.

Corollary 1. *Let $E = \{e_1, \ldots, e_{n_E}\}$ be a given set of states of nature. Let M be an n-item sample matrix, and let Q be an information structure of the plan (n, c); both matrices are defined on the same set of states of nature π. Then there exists a matrix D such that $Q = MD$ and D is an $(n+1) \times 2$ matrix (note: Q is $n_E \times 2$ and M is $n_E \times (n+1)$):*

$$D = \{d_{ij}\}, \qquad d_{ij} = \begin{cases} 1, & i \le c,\ j = 1, \\ & i > c,\ j = 2, \\ 0 & \text{elsewhere} \end{cases}$$

where $0 \le c \le n$.

The Appendix depicts the proof for this corollary.

Results:

1. D is the decision matrix of the sample matrix M, and its structure will be as follows:

$$\begin{array}{r|cc|} & \text{Accept} & \text{Reject} \\ y=0 & 1 & 0 \\ \cdot & 1 & 0 \\ \cdot & 1 & 0 \\ y=c & 1 & 0 \\ y=c+1 & 0 & 1 \\ \cdot & 0 & 1 \\ \cdot & 0 & 1 \\ y=n & 0 & 1 \end{array}.$$

2. A QC plan aggregates the signals (y) into two groups: $y \le c$ and $y > c$. As we have just seen, this aggregation is done by multiplying the sample matrix (M) by the decision matrix (D).
3. The methodology also covers the two degenerated strategies: 'accept always' and 'reject always'. These strategies can be presented by appropriate decision matrices. For instance, the 'accept always' strategy can be presented by the following decision matrix:

$$\begin{vmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \end{vmatrix}.$$

Another important term in QC is the Operating Characteristic Curve (hereafter OCC).

An OCC (Operating Characteristic Curve) is a curve that shows, for a given plan, the probability of acceptance versus the percent of defective parts in the lot. Figure 1 shows an OCC's of two plans (A and, the one whose relations will be discussed later.

One can derive many information structures from a given OCC, depending on the states of nature selected.

4. Ranking QC plans in a 'generally more informative' order

Let Q_A and Q_B be the information matrices of plans A and B, respectively. Q_A will be generally more informative than Q_B if there exists a Markov matrix L that satisfies the equation $Q_B = Q_A L$. This is a result of the Blackwell Theorem (McGuire, 1986).

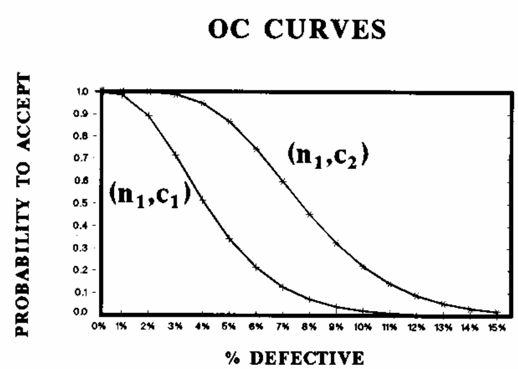

Figure 1. □ Plan A (n_1, c_2) + Plan B (n_1, c_1)

Example 6. Suppose the following two QC plans are given:

$$B = (32, 1), \qquad A = (200, 7).$$

The states of nature are:

$$\text{AQL} = 0.02, \qquad \text{LTPD} = 0.08.$$

The information structures will be:

$$Q_B = \begin{vmatrix} 0.87 & 0.13 \\ 0.26 & 0.74 \end{vmatrix}, \qquad Q_A = \begin{vmatrix} 0.95 & 0.05 \\ 0.08 & 0.92 \end{vmatrix}.$$

A is generally more informative than B because there exists an L,

$$L = \begin{vmatrix} 0.91 & 0.09 \\ 0.21 & 0.79 \end{vmatrix},$$

that satisfies the equation $Q_B = Q_A L$.

This will imply that the expected value of the payoffs gained by using plan A will *never* be less than those obtained by plan B, regardless of the values of the payoff matrix and the a priori probabilities. Thus, plan A *dominates* plan B.

In real-life situations, we usually conduct incoming inspection on items that are intended for use by several users in different projects. Thus, payoff matrices cannot be easily assessed. Having one plan that dominates the other, regardless of the payoff matrix, helps in selecting the right plan. In other cases, where we test a 'brand new' item, for which it is very difficult to estimate the prior probability vector, the choice of a plan that dominates other plans is of prime much importance. It ensures that the test results are valid for a wide range of payoffs and a priori probabilities.

Formally, this order will be defined as follows:

Generally more informative order of QC plans: Plan (n_1, c_1) is generally more informative than plan (n_2, c_2) for a given set of states of nature $(P_1, \ldots, P_{n_E})$ if the QC plan matrix corresponding to (n_1, c_1) is generally more informative than the matrix corresponding to (n_2, c_2).

Theorem 1. *Let (n_1, c_1) and (n_1, c_2) be two QC plans, based on the same sample size (n_1), and $c_2 > c_1$. Neither of the two plans can be considered generally more informative than the other one, for any given set of states of nature.*

Proof. Let $A = (n_1, c_2)$ and $B = (n_1, c_1)$ be two QC plans such that $c_2 > c_1$. Graphically, the OCC of plan A will always be above the OCC of plan B (Juran and Gryna, 1980). Figure 1 shows this relationship.

Let P_1 and P_2 be two states of nature chosen randomly. Let Q_A and Q_B be the plan matrices for A and B. Assume P_1 and P_2 are the states of nature pertaining to this case.

$$Q_A = \begin{matrix} & y \le c_1 & y > c_1 \\ P_1 & 1-\alpha_A & \alpha_A \\ P_2 & \beta_A & 1-\beta_A \end{matrix}, \qquad Q_B = \begin{matrix} & y \le c_1 & y > c_1 \\ P_1 & 1-\alpha_B & \alpha_B \\ P_2 & \beta_B & 1-\beta_B \end{matrix}.$$

Since $c_2 > c_1$ and A is drawn above B,

$$1 - \alpha_A > 1 - \alpha_B \text{ and } \beta_A > \beta_B.$$

Thus, plan A cannot be generally more informative than B, and vice versa, according to the Blackwell Theorem.

Theorem 2. *Let* (n_1, c_1) *and* (n_2, c_1) *be two QC plans having the same acceptance number* (c_1), *and* $n_2 > n_1$. *Neither of the two plans can be considered generally more informative than the other for any set of states of nature. The proof follows the same sequence of reasoning as in the previous theorem.*

Theorem 3. *Let* (n_1, c_1) *and* (n_2, c_2) *be two AC plans such that* $n_1 > n_2$ *and* $c_1 < c_2$, *or one of the inequalities can hold equal. Neither of the two plans can be generally more informative than the other one for any set of states of nature.*

The proof is similar to that in Theorem 1.

Corollary. *For any given QC plans* (n_1, c_1) *and* (n_2, c_2), *a necessary condition that* (n_1, c_1) *will be generally more informative than the other plan is*

$$n_1 > n_2 \text{ and } c_1 > c_2.$$

For example, if we have to choose between the following plans: $A = (250, 5)$, $B = (300, 4)$, there is no way that for any given set of states of nature one plan will be generally more informative than the other, since the conditions stated above ($n_1 > n_2$ and $c_1 > c_2$) do not exist.

So far we have discussed the informativeness of an information structure for a given set of states of nature. The question is whether or not there exists an order that enables a certain plan to be generally more informative than another plan, for any given state of nature. In other words, is it conceivable that a certain plan (n_1, c_1) is generally more informative than an other plan (n_2, c_2) for *all* states of nature? This order ('universally generally more informative') is defined as follows:

Definition. A QC plan (n_1, c_1) is 'universally generally more informative' than (n_2, c_2) if plan (n_1, c_1) is generally more informative than (n_2, c_2) for all states of nature.

We will show now that such a relationship cannot exist.

Theorem 4. *Let* (n_1, c_1) *and* (n_2, c_2) *be QC plans.* (n_1, c_1) *can never be universally generally more informative than* (n_2, c_2) *and vice versa.*

Proof. The relationship between (n_1, c_1) and (n_2, c_2) can be as follows:
(a) *The OCC of* (n_1, c_1) *is always above the OCC of* (n_2, c_2). In this case, according to Theorems 1 and 2, the relationship of generally more informative does not hold.
(b) *The OCCs of* (n_1, c_1) *an* (n_2, c_2) *intersect in the open interval* (0, 1).
Figure 2 portrays this situation.

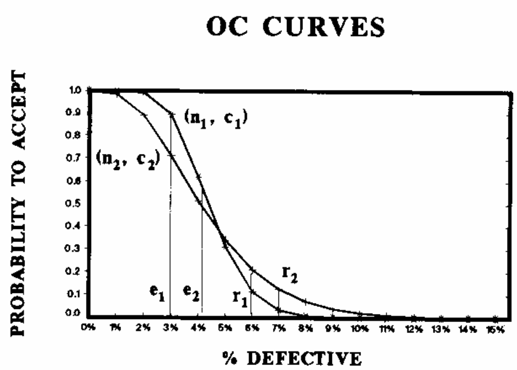

Figure 2.

For the states of nature bounded by the interval $[e_1, e_2]$, the OCC of (n_1, c_1) will always be above that of (n_2, c_2). Therefore (as implied by part (a) of this theorem) the order of universally generally more informative cannot apply. The same argument disables the states of nature bounded by $[r_1, r_2]$ from being universally generally more informative. Since one can always find two states of nature that are in these intervals, the relationship of universally generally more informative does not hold. □

5. Dominance of QC plans having two states of nature

The most common use of QC plans is in the case of two states of nature: AQL and LTPD (Duncan, 1965). The conditions for one QC plan to be generally more informative than the other one is provided by the following theorem:

Theorem 5. *Let Q_A and Q_B be two information structures of QC plans operating on a common set of states of nature* (AQL *and* LTPD). *Let q_{Aij} and q_{Bij} be the elements of these matrices. Q_B is generally more informative than Q_A if and only if*

$$q_{B11} \geq q_{A11} \text{ and } q_{B21} \leq q_{A21}.$$

Proof. According to the Blackwell Theorem (McGuire, 1986), Q_A is generally more informative than Q_B *if and only if* there exists a Markov matrix, L, that satisfies $Q_A L = Q_B$. Thus,

$$\begin{vmatrix} q_{A11} & 1-q_{A11} \\ q_{A21} & 1-q_{A21} \end{vmatrix} \begin{vmatrix} l_{11} & 1-l_{11} \\ l_{21} & 1-l_{21} \end{vmatrix} = \begin{vmatrix} q_{B11} & 1-q_{B11} \\ q_{B21} & 1-q_{B21} \end{vmatrix}.$$

Solving these equations yields

$$l_{11} - l_{21} = \frac{q_{B11} - q_{B21}}{q_{A11} - q_{A21}}$$

since the states of nature are arranged in a descending order $q_{B11} \geq q_{B21}$, and $q_{A11} \geq q_{A21}$. This yields

$$0 \leq l_{11} - l_{21} \quad \text{if } q_{A11} \geq q_{B11} \text{ and } q_{A21} \leq q_{B21},$$

then,

$$0 \leq l_{11} - l_{21} = \frac{q_{B11} - q_{B21}}{q_{A11} - q_{A21}} \leq 1$$

and there are l_{11}, l_{21} that satisfy $0 \leq l_{11} \leq 1$ and $0 \leq l_{21} \leq 1$, which means that L can be presented as a Markov matrix, and the Blackwell Theorem is in effect. □

Interpretations of Theorem 5. Theorem 5 can be interpreted in two ways:

1) *Interpretation by 'error type' considerations.* The matrices Q_A and Q_B can be rewritten as

$$Q_B = \begin{vmatrix} 1-\alpha_{qB} & \alpha_{qB} \\ \beta_{qB} & 1-\beta_{qB} \end{vmatrix}, \qquad Q_A = \begin{vmatrix} 1-\alpha_{qA} & \alpha_{qA} \\ \beta_{qA} & 1-\beta_{qA} \end{vmatrix}$$

where α_{qB}, α_{qA} are the 'producer's risk' of plans Q_B and Q_A, respectively (or, statistically speaking, they are the type I errors).

The condition $q_{A11} \geq q_{B11}$ yields that $\alpha_{qA} \leq \alpha_{qB}$, which means that the 'producer's risk' in Q_A is less than that in Q_B. The second condition $(q_{A21} \leq q_{B21})$ implies that the 'consumer's risk' of plan Q_A is

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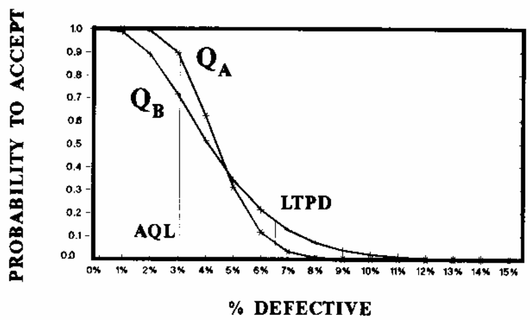

Figure 3.

lower than that of plan Q_B. Thus, Q_A is generally more informative than Q_B if and only if both the 'producer's risk' and the 'consumer's risk' of Q_A are less than those of Q_B, and plan Q_A will always yield a greater expected payoff value than Q_B, regardless of the prior probabilities or the payoff matrix.

2) *Interpretation by OCC.* The graphical interpretation of Theorem 5 is shown in Figure 3.

The relationship of generally more informative between Q_A and Q_B occurs if two conditions are satisfied: (a) the two corresponding OCC's intersect; (b) the point of intersection is between the values of AQL and LTPD.

Result: The relationship of generally more informative can exist only if there are two states of nature. Whenever there exist three or more states of nature, it can be shown that either two or three states are on one side of the intersection. Thus, by Theorem 5 they cannot be rank ordered under the relationship generally more informative.

6. The informativeness of a QC sample and the sample size

Is an $(n + 1)$-item sample lot generally more informative than an n-item one? Intuitively, the bigger the lot, the more information we get. But, is this always true? Take the following case, for example: Assume that you have a 100-item sample, and, a given payoff matrix and prior probabilities. Will a 101-item sample yield more information in the optimal solution?

The following theorem refers to a QC sample, and shows that an $(n + 1)$-item sample is *always* generally more informative than an n-item sample, and the optimal plan (for a given situation) will yield an expected payoff not smaller than the maximal expected payoff derived from an n-size sample. In other words, an $n + 1$ Information Structure of a *sample* is generally more informative than an n IS. (This does not, however contradict Theorem 2 that refers to a QC *plan*).

Theorem 6. *An information structure of an $(n + 1)$-item sample is generally more informative than an information structure of an n-item sample.*

Proof. Let E be a set of states of nature, $E = \{e_1, \ldots, e_{n_E}\}$, and M_{n+1} be an information matrix of an $(n + 1)$-item sample defined on E. Let M_n be an information matrix of an n-item sample defined on E. According to the Blackwell Theorem (McGuire, 1986), M_{n+1} will be generally more informative than M_n

if and only if there exists an $(n+2)\times(n+1)$ Markovian matrix R that satisfies $M_n = M_{n+1} * R$. Using Bayes' Theorem it can be shown that the general element of R, r_{ij}, is given by

$$r_{ij} = \begin{cases} 0, & j > i, \\ 0, & j < i-1, \\ \dfrac{n+2-j}{n+1}, & i = j, \\ \dfrac{j}{n+1}, & j = i-1. \end{cases}$$

$\Sigma r_{ij} = 1$ for every j, so R is a Markov matrix. □

7. Conclusions

This paper deals with the normative value of information derived from QC sampling. The information-economics model was applied to QC problems, showing a practical application of the McGuire model. Terminology was defined to fit QC problems into the model. Terms like information structures of plans and samples, decision matrix of QC plans and samples, payoff matrix and prior vector were defined and analyzed. It was shown that existing terms (such as AQL, LTPD, OCC) can be correlated and used interchangeably with the proposed terminology. The new terminology enables simple and clear application of a Bayesian approach to QC sampling. Theorems on the dominance of one sampling plan over another (in a generally more informative order) were formulated and proved.

Variable sampling cost is not considered in this paper. This important issue calls for further research.

Appendix

Proof of Corollary 1. By definition, Q is an $n_E \times 2$ matrix, and M is an $n_E \times (n+1)$ matrix, using the binomial distribution. Therefore, any D that satisfies the equation $Q = MD$ must be an $(n+1)\times 2$ matrix.

The general element of M (the sample matrix) is

$$m_{ie} = \Pr(y = e-1/P = P_i) = C(n, e-1) * P_i^{e-1} * (1-P_i)^{n-e+1}$$

where $e = 1, \ldots, n+1$.

The general element of Q is

$$q_{i1} = \Pr(y \le c/P = P_i) = \sum_{y=0}^{c} C(n, y) * P_i^y * (1-P_i)^{n-y}.$$

Changing the indices $e = y + 1$ yields

$$q_{i1} = \sum_{e=1}^{c+1} m_{ie} \text{ and } q_{i2} = 1 - \sum_{e-1}^{c+1} m_{ie} = \sum_{e=c+2}^{n+1} m_{ie},$$

and therefore D (as designated above) satisfies $Q = MD$. □

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The Blackwell Theorem (McGuire and Radner, 1986) states that $Q_A \geq Q_B$ if and only if $Q_A L = Q_B$, where L is a Markov matrix with the appropriate dimensions. This ordering is only a partial ordering of the set of finite information structures operating on a given state-of-the-world set.

The gross value of information is always a relative number comparing the expected payoff gained by using different information structures. For example, assuming that the utility is a linear function of the payoff, and Q_B is not better than Q_A, the value of the information of Q_A over Q_B is $F(Q_A, U, \pi) - F(Q_B, U, \pi)$ with an appropriate calibration.

3. Quality control sampling as information structures

We first review some of the important terms of QC sampling, and then incorporate them into the information-structures model. For more details about the quality control terms, the reader is referred to Duncan (1965) or Monks (1982).

3.1. Sample plans and information matrices

A *sample plan* is a decision rule which specifies how large a sample (n) should be and the maximum allowable measurement number, or percentage (c) fo defectives in the sample. A plan is therefore specified by (n, c). For example, the plan (50,2) reads as follows: Select a random sample of 50 units and count the number of defectives. If the number of defectives is equal to or lower than 2, accept the lot; otherwise reject it.

This paper deals with attribute plans, where items are judged dichotomically, as, for example, good or bad, acceptable or rejected. An attribute plan of n units can display $n + 1$ different results (hereafter called signals) which correspond to the possible numbers of defectives identified by the inspection, that is, 0, 1, ,... n defectives. Thus, a QC plan can be regarded as an information structure whose domain is the real quality of a lot (i.e., the percentage of defectives), and whose range is a set of $n + 1$ signals. We shall now formulate this description in a more rigorous definition.

An *information matrix* of a sample M is defined as an information matrix of an n-size sample if it is a Markov matrix as follows:

(1) $M = \{m_{ij}\}$, $i = 1,\dots, n_E$; $j = 1,\dots, n + 1$.

(2) The number of rows is equal to the number of states of nature, which are the possible ratios of defective items, $E = \{P_1,\dots, P_{n_E}\}$.

(3) The number of columns is equal to the number of signals, which are the possible amounts of defective items in the sample $y = 0, 1,\dots, n$. Thus M is an $n_E \times (n + 1)$ matrix as follows:

$$M = \begin{array}{c} P_1 \\ \cdot \\ P_{n_E} \end{array} \left| \begin{array}{c} y=0,\ y=1,\dots,y=n \\ \\ m_{ij} \\ \\ \end{array} \right|$$

(4) $m_{ij} = \Pr(y = j - 1/P = P_i)$, $i = 1,\dots, n_E$; $j = 1,\dots, n + 1$. For convenience, the states of nature will be arranged in an increasing order, so that $P_1 < P_2 < \cdots < P_{n_E}$. This sorting will not change the generality of the problem (see Marschak, 1971).

Example 1. Let M be a sample matrix representing a 3-item sample and suppose there are two possible states of nature: $P_0 = 0.02$, $P_1 = 0.05$. Assuming that the failure probability is binomial (Duncan, 1965), the information system of this sample, M, is

$$M = \begin{array}{c} \\ P_0 = 0.01 \\ P_1 = 0.05 \end{array} \begin{array}{cccc} y=0 & y=1 & y=2 & y=3 \\ \hline 0.9412 & 0.0576 & 0.0012 & 0 \\ 0.8574 & 0.1354 & 0.0071 & 0.0001 \end{array}.$$

For example m_{11} was calculated by using the binomial distribution: $m_{11} = 0.98^3 = 0.9412$.