A Single Bottleneck System with Binomial Yields and Rigid Demand

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This paper considers a "single bottleneck system": a multistage production system where all setups except one are zero. The stage with nonzero setup is defined to be the bottleneck; it may be thought of as the critical resource whose throughput largely determines the throughput of the entire system, as a envisioned by the OPT philosophy. Production is in lots with uncertain (binomial) yields and demand needs to be satisfied in full, thus possibly necessitating multiple production runs. We show how the optimal control problem can be reduced to that of optimal lotsizing a single stage.
(*Random Yield; Binomial Distribution; Rigid Demand; Bottleneck*)

1. Introduction

In considering multistage production systems, the terms bottleneck (BN) or critical resource are often used to describe a resource which is utilized to its full capacity. BNs pace production for the entire system and set an upper bound on performance. An hour lost at a BN is an hour lost for the entire system. These facts have been demonstrated in the findings of recent theoretical and empirical studies, usually associated with the OPT school of thought (e.g., Goldratt and Cox 1986, Lundrigan 1986), suggesting that the determination of optimal lots for the BN, which is the objective of this study, is of utmost importance. Observing that an additional setup on a non-BN (which is idle part of the time) often does not increase operating costs, zero setup costs may be associated with non-BNs (see, for example Ronen and Starr 1990). Therefore, even though other machines/work centers may also have setup costs, the mathematical description we adopt for a system with a single critical resource, is a system where all setup costs but one are zero.

We consider a multistage system where yields in each stage are uncertain, production is costly, and all setup costs but one are zero. We refer to such a system as a single BN system (SBNS) and think of it as a model for a multistage production system with only one critical resource. We study a SBNS with binomial yields and rigid demand; that is, there is a constant probability that an item exiting a machine is defective, and demand needs to be satisfied in full. Defectives have no value and only good parts may proceed from one stage to another. If at the end of production the number of good products exceeds the demand, the excess has no value. This situation often arises in high-tech industries, where many products are custom-made. The tradeoff here is between using small lots, possibly involving repeat setups, and using large lots, which may result in unnecessary and costly overproduction.

In the context of random yield and rigid demand, many authors have assumed that there is no salvage value (e.g., Beja 1977, Grosfeld-Nir and Gerchak 1990, Sepheri et al. 1986, Spence 1988, White 1965 and references therein). One of the reasons is that typically the setup cost is high and the quantity ordered is small (if the setup cost is low or the quantity ordered is large, then the relative cost of setups is negligible and the simple policy "produce the quantity ordered" is quite satisfactory). This description fits make-to-order production where, very often, the excess has no value. Assuming nonzero salvage value complicates the computations considerably, because probabilities for all possible sizes of excess production must be computed.

0025-1909/93/3905/0650$01.25

Also, typically, the excess of usable products is small, and thus does not much affect the optimal solution obtained by the zero salvage cost assumption. Intuitively, we wish to propose that nonzero salvage costs will slightly reduce the production cost and increase the optimal lot size.

Lotsizing in single-stage production systems with random yield and nonrigid demand is now well understood, both for binomial yields (e.g., Levitan 1960) and stochastically proportional yields[1] (e.g., Gerchak et al. 1988). When demand is rigid, single-stage systems have been solved via dynamic programming and heuristically (e.g., Klein 1966, Beja 1977, Sepheri et al. 1986). A multistage system with nonrigid demand has been modeled by Lee and Yano (1988). Spence (1988) provided the only attempt to model multistage systems with rigid demand. She assumed that, in each stage, defective units can be perfectly reworked, thus necessitating at most two runs at each stage. This paper does not make any such assumptions and thus constitutes the first attempt to deal with an unbounded random number of runs for a multistage system with rigid demand. A full-blown analysis of such systems, allowing for setups at each stage, seems to be an unrealistic goal at the present time. Assuming the presence of a single machine with a nonzero setup cost is thus natural and is also consistent with the single critical resource model of OPT.

We refer to a system where *all* setup costs are zero as a zero-BN system (0-BNS). When a 0-BNS faces rigid demand, it is optimal to process parts *one-by-one* until the demand is satisfied. The SBNS may be viewed as consisting of three components: two 0-BNS and the BN (Figure 1). The objective is to determine an optimal policy which minimizes the expected cost of satisfying the demand. Clearly, the optimal policy has the following structure: parts should be processed *one-by-one* by the first 0-BNS, until a certain size lot of good parts is ready to be processed by the BN. Then, the lot (as a whole) enters the BN and finally, the usable parts which exit the BN are processed *one-by-one* by the second 0-BNS, until demand is satisfied or all parts are exhausted. If, at the end of production, demand is not yet satisfied, further production runs must be initiated as necessary. Thus, the optimal policy is completely characterized once the optimal lot for the SBNS (the optimal lot for the BN) is determined.

[1] The yields are said to be stochastically proportional if for any lot Q, the yield Y_Q satisfies $Y_Q = QX$, where X is a random variable independent of Q.

2. The Model

We denote by $[\alpha, \beta, \theta]$ a machine with fixed setup cost α, constant processing cost per unit β and success probability θ.

We assume the following scheduling rule. A single unit is processed sequentially on each of the first $(k-1)$ machines until it fails at a given stage or is completed as a usable unit ready to be processed by the BN. As soon as the processing of a unit is completed a new unit is mounted onto the first 0-BNS, until a certain batch size of good units has been processed by the first 0-BNS. Then the batch enters the BN, and a single unit, out of the usable units exiting the BN, is processed sequentially on each of the remaining $(M-k)$ machines, until it fails at a given stage or is completed as a final product; as soon as the processing of a unit is completed, a new unit is mounted onto the second 0-BNS, until D successful final units are obtained or all units have been processed. (The first and second 0-BNSs thus process at most one unit at a time.) Note that this scheduling policy minimizes variable processing costs.

Some reflection reveals that the SBNS of Figure 1 can be reduced to the following two-stage SBNS:

$$\rightarrow [\hat{\alpha}_1, \hat{\beta}_1, \hat{\theta}_1] \rightarrow [0, \hat{\beta}_2, \hat{\theta}_2] \rightarrow D \quad \text{where}$$

$$\hat{\alpha}_1 = \alpha_k, \quad \hat{\theta}_1 = \theta_k, \quad \text{and} \tag{1}$$

$$\hat{\beta}_1 = \frac{\beta_1}{\theta_1 \cdots \theta_{k-1}} + \frac{\beta_2}{\theta_2 \cdots \theta_{k-1}} + \cdots + \frac{\beta_{k-1}}{\theta_{k-1}} + \beta_k, \tag{2}$$

$$\hat{\beta}_2 = \beta_{k+1} + \theta_{k+1}\beta_{k+2} + \theta_{k+1}\theta_{k+2}\beta_{k+3} + \cdots + \theta_{k+1} \cdots \theta_{M-1}\beta_M, \tag{3}$$

$$\hat{\theta}_2 = \theta_{k+1}\theta_{k+2} \cdots \theta_M. \tag{4}$$

This two-stage system can be further reduced to a single stage with a setup cost $\alpha = \hat{\alpha}_1$, a variable processing cost $\beta_D(N)$ (associated with a lot of size N and demand D), and a "truncated" negative binomial yield pattern. That is

Figure 1 **An M-machine SBNS consists of two 0-BNS and the BN. The BN (the kth machine) has setup cost α_k**
The parameters β_i and θ_i are the processing cost and the success probability of the ith machine, respectively. The (rigid) demand is D.

$$\rightarrow [0, \beta_1, \theta_1] \rightarrow \cdots \rightarrow [0, \beta_{k-1}, \theta_{k-1}] \rightarrow [\alpha_k, \beta_k, \theta_k] \rightarrow [0, \beta_{k-1}, \theta_{k+1} \rightarrow \cdots \rightarrow [0, \beta_M, \theta_M] \rightarrow D$$

first 0-BNS — BN — second 0-BNS

$$\beta_D(N) = N\hat{\beta}_1 + \hat{\beta}_2 \left\{ \sum_{L=0}^{D-1} L \binom{N}{L} \hat{\theta}_1^L (1 - \hat{\theta}_1)^{N-L} + \sum_{L=D}^{N} \binom{N}{L} \hat{\theta}_1^L (1 - \hat{\theta}_1)^{N-L} \times \left[\sum_{x=D}^{L} x \binom{x-1}{D-1} \hat{\theta}_2^D (1 - \hat{\theta}_2)^{x-D} + L \sum_{x=L+1}^{\infty} \binom{x-1}{D-1} \hat{\theta}_2^D (1 - \hat{\theta}_2)^{x-D} \right] \right\}. \quad (5)$$

L represents the number of units in the lot which pass the BN-stage. The number of units processed on the second 0-BNS is L, if $L < D$; if $L \geq D$, it is a negative binomial variable truncated at L. Also, let $\theta = \hat{\theta}_1 \hat{\theta}_2$; then, the yield associated with a lot N is

$$p(x, N) = \begin{cases} \binom{N}{x} \theta^x (1 - \theta)^{N-x}, & x = 0, 1, \ldots, D-1, \\ 1 - \sum_{x=0}^{D-1} \binom{N}{x} \theta^x (1 - \theta)^{N-x}, & x = D. \end{cases} \quad (6)$$

Let F_D be the minimal expected cost to fulfill an order D. Then, the optimal lot and optimal expected cost can be computed recursively in D:

$$F_D = \min_{N \geq D} \left\{ \alpha + \beta_D(N) + \sum_{x=0}^{D-1} \binom{N}{x} \theta^x (1 - \theta)^{N-x} F_{D-x} \right\} = \min \left\{ \frac{\alpha + \beta_D(N) + \sum_{x=1}^{D-1} \binom{N}{x} \theta^x (1 - \theta)^{N-x} F_{D-x}}{1 - (1 - \theta)^N} \right\}. \quad (7)$$

The algorithm is very efficient; using a personal computer it takes just a few seconds to compute an optimal lot size for D as high as 200.

3. Optimal Sequencing

Let m be the expected cost of producing one good product by an M-machine 0-BNS (set $\alpha_k = 0$ in Figure 1). Then

$$m = \beta_1 + \theta_1 \beta_2 + \theta_1 \theta_2 \beta_3 + \cdots + \theta_1 \cdots \theta_{M-1} \beta_M + (1 - \theta_1 \cdots \theta_M) m. \quad (8)$$

This leads to

$$m = \frac{\beta_1}{\theta_1 \cdots \theta_M} + \frac{\beta_2}{\theta_2 \cdots \theta_M} + \cdots + \frac{\beta_M}{\theta_M}. \quad (9)$$

Therefore, the expected cost of producing D good products by the 0-BNS is Dm.

We now explain how to arrange the machines if altering their order is permitted. We start with the optimal sequencing of a 0-BNS.

3.1. Proposition. *The 0-BNS produces with minimal expected cost if the machines are arranged so that the ratio $\beta_i/(1 - \theta_i)$ is increasing.*

For a proof see Lee (1988).

3.2. Corollary. *Referring to Figure 1, suppose that the order of the machines of the first (second) 0-BNS may be altered. Then, the best sequence is obtained by the rule of the last proposition.*

3.3. Corollary. *Referring to Figure 1, the overall optimal sequence may be determined by solving the dynamic program (8) $M2^{M-1}$ times, once for every possible position combination of the* BN *and the set of machines on the second* 0-BNS.

4. Concluding Remarks

(*a*) We have shown how to reduce the lotsizing problem of a SBNS to that of a single stage, which is of great importance to users of the OPT approach. We have used binomial yields; it is of interest to verify whether similar results can be obtained for other yield patterns as well.

(b) As we have mentioned, we do not know how to solve the problem of a system where more than one stage is allowed to have nonzero setup. Moreover, we can not solve even the simplest problem:

$$\rightarrow [\alpha_1, \beta_1, \theta_1] \rightarrow [\alpha_2, \beta_2, \theta_2] \rightarrow D = 1,$$

namely, the two-machine system facing the rigid demand of one unit. The source of the difficulty is that for each level of inventory (usable output of the first machine) one of the two following actions must be selected: "continue" (produce on the second machine) or "return" (produce more units on the first machine). Though it is conceivable that the optimal rule has the form "continue if and only if the inventory level exceeds some control limit", the following dilemma still remains: if "continue" is optimal, how many units should be processed on the first machine and if "return" is optimal, how many units should be processed on the first machine? Mathematically speaking, equations for a certain level of inventory involve terms depending upon optimal costs for yet unknown levels of inventory, making a recursive solution impossible.

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Accepted by Awi Federgruen; received June 28, 1990. This paper has been with the authors 13 months for 4 revisions.